

# A Brief Note on Foliations of Constant Gaussian Curvature

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**Abstract:** This note provides an alternative proof of a result of Labourie. We show that the two complements of the convex core of a three dimensional quasi-fuchsian hyperbolic manifold may be foliated by embedded hypersurfaces of constant Gaussian curvature.

**Key Words:** foliation, quasi-fuchsian, hyperbolic geometry.

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## 1 - Introduction.

Let  $M$  be a geometrically finite 3 dimensional hyperbolic manifold without cusps. In [1], Labourie proved that the complement of the convex core of such a manifold may be foliated by surfaces of constant Gaussian curvature  $k$  taking values in the interval  $]0, 1[$ . This permits the construction of a smooth family of submersions parametrised by  $k$  from the moduli space of  $M$  into the Cartesian product of finitely many Teichmüller spaces. In this letter, we present a proof of this result in the simpler case where  $M$  is a quasi-Fuchsian manifold which may be easily be generalised to higher dimensions.

Let  $\mathbb{H}^3$  be 3 dimensional hyperbolic space. We recall that the ideal boundary of  $\mathbb{H}^3$  may be identified with the Riemann sphere. Let  $\Gamma$  be a cocompact Fuchsian group and let  $\rho : \Gamma \rightarrow \text{Isom}(\mathbb{H}^3)$  be a quasi-fuchsian representation. The quotient space  $M = \mathbb{H}^3 / \rho(\Gamma)$  is a non compact hyperbolic manifold. The limit set of  $\rho(\Gamma)$  in  $\partial_\infty \mathbb{H}^3 = \hat{\mathbb{C}}$  is a Jordan curve  $c$  which is invariant under the action of  $\rho(\Gamma)$ . Let  $C$  be the convex hull of  $c$  in  $\mathbb{H}^3$ . This is also invariant under the action of  $\rho\Gamma$  and thus quotients down to a compact convex subset of  $M$ , which we will refer to as the convex core of  $M$  and will also denote by  $C$ . The complement of the convex core consists of two open non-compact connected components.

The complement in  $\hat{\mathbb{C}}$  of the curve  $c$  consists of two invariant simply connected domains. Let  $\Omega$  be one of these domains.  $\Omega$  is the interior of the ideal boundary of one of the connected components of the complement of  $C$  in  $\mathbb{H}^3$ . Let us denote this component by  $U$ .  $\Omega$  defines a Plateau problem in the sense of Labourie [2]. Thus, using results of Labourie, [2], or Rosenberg and Spruck, [3], for all  $k \in ]0, 1[$ , there exists a unique convex embedded submanifold  $\Sigma_k = (S, i_k)$  of constant Gaussian curvature equal to  $k$  such that if  $\hat{i}_k$  is the Gauss lifting of  $i_k$  and if  $\overrightarrow{n} : U\mathbb{H}^3 \rightarrow \hat{\mathbb{C}}$  is the Gauss Minkowski mapping, then  $\varphi_k = \overrightarrow{n} \circ \hat{i}_k : S \rightarrow \Omega$  is a homeomorphism. By [4], the immersion  $i_k$  varies continuously with  $k$ . Moreover, the ideal boundary of  $\Sigma_k$  in  $\partial_\infty \mathbb{H}^3$  coincides with  $c$ .

We obtain the following result:

### Theorem 1.1

*$(\Sigma_k)_{k \in ]0, 1[}$  defines a smooth foliation of  $U$  invariant under the action of  $\rho\Gamma$ . Moreover,  $\Sigma_k$  converges towards  $\Omega$  and  $\partial U = \overline{U} \cap \overline{C}$  in the Hausdorff topology in  $\mathbb{H}^3 \cup \partial_\infty \mathbb{H}^3$  as  $k$  tends towards 1 and 0 respectively.*

**Proof:** This result follows directly from the lemmata that we will establish in the following sections. Firstly, by Lemma 3.1,  $(\Sigma_k)_{k \in ]0, 1[}$  is a continuous foliation of some subset of  $U$ . Lemmata 4.1 and 4.4 allow us to establish the convergence properties of this foliation as  $k$  tends towards 1 and 0 respectively. This allows us in turn to show that the foliation covers the whole of  $U$ . Finally, by Lemma 5.1, the foliation is smooth.  $\square$

By uniqueness, for all  $k$ ,  $\Sigma_k$  is invariant under the action of  $\rho(\Gamma)$ . Consequently, this foliation projects down to a smooth foliation of one of the connected components of the complement of the convex core of  $M = \mathbb{H}^3 / \rho(\Gamma)$ , having analogous convergence properties as  $k$  tends to 1 and 0.

Since the result of Rosenberg and Spruck is the only active ingredient of the proof, this result also holds in higher dimensions.

The study of the geometry of this foliation presents many interesting questions. Firstly, we would like to know how the distance from the surface  $\Sigma_k$  to the convex core varies with  $k$ . Indeed, by the proof of Lemma 4.4, we see that this distance is bounded above by  $\operatorname{arctanh}(\sqrt{k})$ . Moreover, by considering the Fuchsian case, where  $\rho\Gamma \subseteq \mathbb{PSL}(2, \mathbb{R})$  and the convex core is a totally geodesic submanifold, we see that this bound is strict. However, following the proof of Lemma 4.1, a bound from below is more subtle. One would anticipate, however, that such a bound may be obtained as a function of the complex dilatation or Schwarzian derivative of  $\Omega$ , or of the Hausdorff dimension of  $c$ .

Secondly, we would like to study the behaviour of the area of  $\Sigma_k$  and the volume contained between  $\Sigma_k$  and the convex core as  $k$  tends towards each of the limits. This corresponds to the study of the asymptotic behaviour of the integrals of 1 and  $f$  over  $\Sigma_k$  as  $k$  tends to 0 and 1. In particular, the constant terms in the asymptotic expansions of these functions would yield a renormalised area and volume.

Thirdly, we observe that an analogous construction may be carried out in the case of pointed  $k$ -surfaces, as in [5], although in this case we obtain a lamination rather than a foliation. In particular, in [6], we show that the area and the volume of pointed  $k$ -surfaces are well defined and finite. Thus analogous geometrical questions may also be studied in this case.

As a final observation, the immersed hypersurface  $\Sigma_k$  carries two different canonical metrics, being the pull back through  $i$  of the hyperbolic metric of  $\mathbb{H}^3$ , the pull back through  $\hat{i}$  of the metric of  $U\mathbb{H}^3$ . It also carries a canonical conformal structure, being the pull back through  $\overrightarrow{n} \circ \hat{i}$  of the conformal structure of  $\hat{\mathbb{C}}$ . Using, for example, results proven in [5], one may show that the Teichmüller distance separating these three conformal structures converges to zero as  $k$ .

This paper is arranged as follows. In the following section, we calculate the effect of a normal deformation on the Gaussian curvature of a surface, and in the remaining sections, we prove the different parts of Theorem 1.1.

## 2 - Normal Deformation of Immersed Surfaces.

Let  $M$  be a three dimensional hyperbolic manifold and let  $\Sigma = (S, \varphi)$  be an immersed hypersurface. We begin by calculating the effect on the second fundamental form of  $\Sigma$  of a variation in the normal direction.

Let  $\Phi : S \times ]-\epsilon, \epsilon[ \rightarrow M$  be an isotopy of immersions. That is, for all  $t \in ]-\epsilon, \epsilon[$ ,  $\Phi_t = \Phi(\cdot, t)$  is an immersion. Suppose that  $\Phi_0 = \varphi$ . Suppose moreover, that the variation is normal. That is,  $\partial_t$  is orthogonal to the tangent space to  $S$  with respect to the induced metric obtained by pulling back the hyperbolic metric on  $M$ . Let  $N$  be the vector field over  $\Phi$  such that, for all  $t$ ,  $N_t = N(\cdot, t)$  is the exterior unit normal vector field to  $\Sigma_t = (S, \Phi_t)$ . Let  $f_t : S \rightarrow \mathbb{R}$  be such that:

$$\Phi_* \partial_t = f_t N_t.$$

Finally, for all  $t$ , let  $A_t$  be the Weingarten operator of  $\Sigma_t$ . Thus, for all  $X$  in  $TS$ :

$$A \cdot X = \nabla_X \mathbf{N}.$$

We obtain the following result:

**Lemma 2.1**

*The first derivative of  $A$  is given by:*

$$\partial_t A = f \text{Id} - \text{Hess}(f) - f A^2.$$

**Proof:** Since this is a local formula, we may assume that  $\Phi$  is a diffeomorphism. We thus identify  $M$  with  $S \times ]-\epsilon, \epsilon[$  and  $\Phi$  with the identity.

The first step involves calculating  $\nabla_{\partial_t} \mathbf{N}$ . Since the norm of  $\mathbf{N}$  is constant, we have:

$$\langle \nabla_{\partial_t} \mathbf{N}, \mathbf{N} \rangle = 1/2 \partial_t \langle \mathbf{N}, \mathbf{N} \rangle = 0.$$

Likewise, if  $X$  is tangent to  $S$ ,  $\langle \mathbf{N}, X \rangle$  always vanishes, and thus:

$$\langle \nabla_{\partial_t} \mathbf{N}, X \rangle = -\langle \mathbf{N}, \nabla_{\partial_t} X \rangle = -\langle \mathbf{N}, \nabla_X f \mathbf{N} \rangle.$$

Since  $\nabla_X \mathbf{N}$  is tangent to  $S$ , this yields:

$$\langle \nabla_{\partial_t} \mathbf{N}, X \rangle = -\langle \nabla f, X \rangle.$$

It thus follows that:

$$\nabla_{\partial_t} \mathbf{N} = -\nabla^\Sigma f.$$

Where  $\nabla^\Sigma f$  is the projection onto  $T\Sigma$  of the gradient of  $f$ . Now, for all  $X, T \in TS$ , we have:

$$\begin{aligned} \partial_t \langle \nabla_X \mathbf{N}, Y \rangle &= \langle \nabla_{\partial_t} \nabla_X \mathbf{N}, Y \rangle + \langle \nabla_X \mathbf{N}, \nabla_{\partial_t} Y \rangle \\ &= \langle R_{\partial_t X} \mathbf{N}, Y \rangle + \langle \nabla_X \nabla_{\partial_t} \mathbf{N}, Y \rangle + \langle \nabla_X \mathbf{N}, \nabla_Y f \mathbf{N} \rangle. \\ &= \langle f R_{\mathbf{N}X} \mathbf{N}, Y \rangle - \langle \nabla_X \nabla^\Sigma f, Y \rangle + \langle A \cdot X, A \cdot Y \rangle. \end{aligned}$$

Observe that the above formula contains not only the variation of  $A$  but also the variation of the metric. We thus have to subtract the variation of the metric in order to obtain the variation of  $A$ . For each  $t$ , we denote by  $g_t$  the restriction of  $\langle \cdot, \cdot \rangle$  to the surface  $S_t = S \times \{t\}$ .

We have:

$$\begin{aligned} \partial_t g_t(X, Y) &= g_t(\nabla_{\partial_t} X, Y) + g_t(X, \nabla_{\partial_t} Y) \\ &= g_t(\nabla_X \partial_t, Y) + g_t(X, \nabla_Y \partial_t) \\ &= f g_t(A \cdot X, Y) + f g_t(X, A \cdot Y). \end{aligned}$$

Combining this with the preceding formula, and bearing in mind that  $A$  is symmetric, we obtain:

$$g_t((\partial_t A_t) \cdot X, Y) = g_t(f R_{\mathbf{N}X} \mathbf{N} - \text{Hess}(f) \cdot X - f A^2, Y).$$

Finally, since  $M$  is hyperbolic, for all  $X$  tangent to  $S$ , the curvature satisfies:

$$R_{NX}N = X.$$

The result now follows.  $\square$

In particular, when  $f \cong 1$ , we obtain the family of immersed surfaces equidistant from  $\Sigma$ . Let us note by  $\Sigma_t = (S, i_t)$  this family, and for all  $t$  let  $A_t$  be the Weingarten operator of  $\Sigma_t$ . We obtain the following corollary:

**Corollary 2.2**

*$A_t$  satisfies the following differential equation:*

$$\partial_t A_t = \text{Id} - A_t^2.$$

This permits us to calculate  $A_t$  over all time. Indeed, let  $\lambda_1(t) \leq \lambda_2(t)$  be the two eigenvalues of  $A_t$ . We have the following result:

**Corollary 2.3**

*For each  $i$ , there exists a constant  $T_0 \in \mathbb{R}$  such that, either  $\lambda_i(t) = \tanh(T_0 + t)$  or  $\lambda_i(t) = \cotanh(T_0 + t)$  or  $\lambda_i(t) = 1$ .*

**Proof:** Let  $p$  be a point in  $S$ . Let  $E_1, E_2$  be the principal eigenvectors of  $A_0$  at  $p$ . If we now define the symmetric matrix  $\hat{A}_t$  such that  $E_1$  and  $E_2$  are its eigenvectors and its eigenvalues are given by the formulae in the statement of this corollary, then  $\hat{A}_t$  satisfies the same differential equation as  $A_t$  with the same initial conditions, and they thus coincide. The result now follows.  $\square$

This permits us to obtain the following bound from above:

**Lemma 2.4**

*Let  $k \in ]0, 1[$  and suppose that  $\Sigma$  is convex and of constant Gaussian curvature equal to  $k$ . Suppose, moreover, that there exists  $a > 0$  such that, in the sense of positive definite matrices:*

$$A_0 \geq a \text{Id}.$$

*Then, there exists a function  $K : [0, +\infty[ \rightarrow [0, 1]$  which only depends on  $k$  and  $a$  such that:*

- (i)  $K$  is strictly increasing,
- (ii)  $K(t)$  tends to 1 as  $t$  tends to  $+\infty$ , and
- (iii) for all  $t$ , the Gaussian curvature of  $\Sigma_t$  is bounded above by  $K(t)$ .

**Proof:** Let  $p$  be a point in  $S$ . For all  $t$ , let  $\lambda_1(t) \leq \lambda_2(t)$  be the eigenvalues of  $A_t(p)$ . Since  $\lambda_1 > 0$  and  $0 < \lambda_1 \lambda_2 < 1$ , by Corollary 2.3, there exists  $A_1 > \hat{a} := \operatorname{arctanh}(a)$  such that:

$$\lambda_1(t) = \tanh(A_1 + t).$$

Corollary 2.3 now offers three possibilities for the evolution of  $\lambda_2$ , depending on the value of  $A_1$ :

(i) If  $A_1 < \operatorname{arctanh}(k)$  then  $\lambda_1(0) < k$  and so  $\lambda_2(0) > 1$ . Thus, there exists  $A_2 > 0$  such that:

$$\lambda_2(t) = \operatorname{cotanh}(A_2 + t).$$

Since  $\lambda_1 \lambda_2 < 1$ ,  $A_2 > A_1$ . Consider the function  $\varphi(t) = \operatorname{cotanh}(c + t) \tanh(t)$  where  $c > 0$ . Using elementary hyperbolic trigonometry, we obtain:

$$\partial_t \varphi(c, t) = \sinh(2c)/2(\cosh(t) \sinh(c + t))^2 > 0.$$

Consequently,  $\varphi(c, t)$  is increasing in  $t$ . Moreover,  $\varphi(c, t)$  tends to 1 as  $t$  tends to infinity. Since  $\varphi((A_2 - A_1) + A_1, A_1) = k < 1$  and  $A_1 > \hat{a}$ :

$$\begin{aligned} \varphi(A_2 - A_1 + \hat{a}, \hat{a}) &\leq k \\ \Rightarrow A_2 - A_1 &\geq \operatorname{arccotanh}(ka) - \hat{a} \\ \Rightarrow A_2 &\geq \operatorname{arccotanh}(ka). \end{aligned}$$

We now define  $\hat{A}_1(a, k) = \operatorname{arctan}(k)$  and  $\hat{A}_2(a, k) = \operatorname{arccotanh}(ka)$ . We define  $K_1(a, k; t) : [0, \infty[ \rightarrow \mathbb{R}$  by:

$$\begin{aligned} K_1(a, k; t) &= \tanh(\hat{A}_1(a, k) + t) \operatorname{cotanh}(\hat{A}_2(a, k) + t) \\ &= \varphi(\hat{A}_2(a, k) - \hat{A}_1(a, k), t + \hat{A}_1(a, k)). \end{aligned}$$

The function  $K_1$  is increasing in  $t$  and tends to 1 as  $t$  tends to infinity. Moreover, since  $\tanh$  is increasing and  $\operatorname{cotanh}$  is decreasing, bearing in mind the bounds on  $A_1$  and  $A_2$ , for all  $t$ :

$$K_1(a, k; t) \geq \lambda_1(t) \lambda_2(t).$$

$K_1$  is thus the desired function in this case.

(ii) If  $A_1 = \operatorname{arctanh}(k)$ , then  $\lambda_1(0) = k$  and so  $\lambda_2(0) = 1$ . Thus, for all  $t$ :

$$\lambda_2(t) = 1.$$

We define  $K_2(a, k; t) = \tanh(\operatorname{arctanh}(k) + t)$ , and we see that  $K_2$  is the desired function in this case.

(iii) If  $A_2 > \operatorname{arctanh}(k)$ , then  $\lambda_1(0) > k$  and so  $\lambda_2(0) < 1$ . Thus, there exists  $A_2 > A_1$  such that:

$$\lambda_2(t) = \tanh(A_2 + t).$$

We define  $\hat{A}_1 = \operatorname{arctanh}(\sqrt{k})$  and  $\hat{A}_2 = \operatorname{arctanh}(k/a)$ . We have:

$$A_1 \leq \hat{A}_1, \quad A_2 \leq \hat{A}_2.$$

We define the function  $K_3(a, k; t)$  by:

$$K_3(a, k; t) = \tanh(\hat{A}_1 + t) \tanh(\hat{A}_2 + 2).$$

We see that  $K_3$  is the desired function in this case. The result now follows by taking the maximum of  $K_1$ ,  $K_2$  and  $K_3$ .  $\square$

### 3 - The Family is a Foliation.

#### Lemma 3.1

For  $k \neq k'$ ,  $\Sigma_k$  and  $\Sigma_{k'}$  are disjoint.

**Proof:** We may assume that  $k > k'$ . For  $t \in [0, 1]$  let  $\Omega_t$  be a family of Jordan subdomains of  $\Omega$  such that:

- (i)  $\Omega_0$  is a disc,
- (ii) for  $t < t'$  the closure of  $\Omega_t$  is contained in  $\Omega_{t'}$ , and
- (iii)  $\partial\Omega_t$  tends to  $c = \partial\Omega$  as  $t$  tends to 1.

Such a family may be obtained, for example, by uniformising the annulus obtained by removing a disc from  $\Omega$ . Let  $\delta$  be a real number such that  $0 < \delta < 1 - k$ . For  $t \in [0, 1]$  we define  $k_t \in [k, 1]$  by:

$$k_t = (1 - \delta)(1 - t) + kt.$$

For all  $t$ , let  $\Sigma'_t = (S, j_t)$  be the unique solution to the Plateau problem given by  $\Omega_t$  with constant Gaussian curvature equal to  $k_t$ . We may assume that we have chosen  $\Omega_0$  sufficiently small so that  $\Sigma'_0$  is disjoint from  $\Sigma_{k'}$  in  $\mathbb{H}^3 \cup \partial_\infty \mathbb{H}^3$ .

We define  $t_0 \in [0, 1]$  by:

$$t_0 = \inf\{t \in [0, 1] \text{ s.t. } \Sigma'_t \cap \Sigma_{k'} \neq \emptyset \text{ in } \mathbb{H}^3 \cup \partial_\infty \mathbb{H}^3\}.$$

Suppose that this set is non-empty, and thus that  $t_0 \in [0, 1]$ . By compactness,  $\Sigma'_{t_0}$  and  $\Sigma_{k'}$  intersect non-trivially in  $\mathbb{H}^3 \cup \partial_\infty \mathbb{H}^3$ . Since the ideal boundaries of  $\Sigma'_{t_0}$  and  $\Sigma_{k'}$  are  $\partial\Omega_{t_0}$  and  $\partial\Omega$  respectively, and since these do not intersect, it follows that these two surfaces intersect non-trivially in  $\mathbb{H}^3$ . However, by continuity, since  $\Sigma_0$  lies in (the closure of) the exterior of  $\Sigma_{k'}$ , so does  $\Sigma'_{t_0}$ . Consequently,  $\Sigma'_{t_0}$  is an exterior tangent to  $\Sigma_{k'}$ . This is absurd by the geometric maximum principal, since the Gaussian curvature of  $\Sigma'_{t_0}$  exceeds that of  $\Sigma_{k'}$ . Consequently, the set used to define  $t_0$  is empty, and the result now follows.  $\square$

### 4 - Completeness of the Foliation.

We recall that  $\mathbb{H}^3 \cup \partial_\infty \mathbb{H}^3$  has the topology of a closed ball. We now prove completeness of the foliation near infinity:

#### Lemma 4.1

$(\Sigma_k)_{k \in [0, 1]}$  tends towards  $\Omega$  in the Hausdorff topology as  $k$  tends to infinity.



**Proof:** For all  $k$ , let  $A_k$  be the Weingarten operator of  $\Sigma_k$ . Let  $k_0 \in [0, 1]$  be arbitrary. Since  $\rho(\Gamma)$  has a cocompact action of  $\Sigma_{k_0}$ , there exists  $a > 0$  such that, in the sense of positive definite symmetric matrices, for all  $p \in S$ :

$$A_k(p) \geq a \text{Id}.$$

Let the function  $K(a, k_0; \cdot) : [0, +\infty[ \rightarrow \mathbb{R}$  be as in Lemma 2.4. For all  $d \in [0, \infty[$ , let  $\Sigma'_d = (S, j_d)$  be equidistant surface of distance  $d$  from  $\Sigma_{k_0}$ . For all  $d$ , let  $k'_d : S \rightarrow ]0, \infty[$  be the Gaussian curvature of  $\Sigma'_d$ . By Lemma 2.4, for all  $d$ , and for all  $p \in S$ :

$$k'_d(p) \leq K(a, k_0; d) < 1.$$

Let  $D > 0$  be arbitrary. By a continuity argument identical to that employed in the proof of Lemma 3.1, for all  $k \geq K(a, k_0; D)$ , the immersed surface  $\Sigma_k$  lies in the exterior of  $\Sigma'_D$ . Since  $D$  may be chosen arbitrarily large, the result now follows.  $\square$

In order to prove the completeness of the foliation near the convex core, we require the following definition of Gaussian curvature in the weak sense:

**Definition 4.2**

Let  $(N, i)$  be a (not necessarily  $C^2$ ) hypersurface in a manifold  $M$ . Let  $p$  be a point in  $N$ . We say that  $(N, i)$  is weakly convex at  $p$  if there exist open neighbourhoods  $U \subseteq N$ ,  $V \subseteq M$  of  $p$  and  $i(p)$  respectively and a convex subset  $K \subseteq V$  such that  $i(p) \in \partial K$  and, for all  $q \in U$ ,  $i(q) \in K$ .

If  $(N, i)$  is weakly convex at  $p$ , for  $k > 0$ , we say that the Gaussian curvature of  $(N, i)$  is at least  $k$  in the weak sense if and only if  $U$ ,  $V$  and  $K$  may be chosen such that  $\partial K$  is smooth and that the Gaussian curvature of  $K$  at  $i(p)$  is equal to  $k$ .

For all  $d$ , let  $\Sigma'_d = (S, j_d)$  be the equidistant surface in  $U$  of distance  $d$  from the convex core. We have the following result:

**Lemma 4.3**

For all  $d \in [0, +\infty[$  and for all  $p \in S$ ,  $\Sigma'_d$  is weakly convex at  $p$  and its Gaussian curvature is at least  $\tanh(d)^2$  in the weak sense.

**Proof:** Let  $d$  be a non negative real number. Let  $p$  be a point in  $S$ . Let  $q$  be the point in  $\partial U$  closest to  $i_d(p)$ . There exists a supporting hyperplane to  $C$  at  $q$  whose normal points towards  $i_d(p)$ . Let  $P$  be this supporting hyperplane. Let  $P_d$  be the equidistant surface in  $U$  at distance  $d$  from  $P$ . Since  $C$  lies entirely to one side of  $P$ ,  $\Sigma'_d$  lies within the interior of  $P_d$ . Moreover, since the normal of  $P$  at  $q$  points towards  $i_d(p)$ ,  $i_d(p)$  lies on  $P_d$ . However,  $P_d$  is strictly convex, and, by Corollary 2.3, its Gaussian curvature equals  $\tanh(d)^2$ . The result now follows.  $\square$

We are now in a position to prove the completeness of the foliation near the convex core:

**Lemma 4.4**

$(\Sigma_k)_{k \in [0, 1]}$  tends towards  $\partial U$  in the Hausdorff topology as  $k$  tends to 0.

**Proof:** Let  $\Omega'$  be the other connected component of  $c$  in  $\hat{C}$ . For  $t \in [0, 1]$ , let  $\Omega'_t$  be a family of Jordan subdomains of  $\Omega'$  such that:

- (i)  $\Omega'_0$  is a disc,
- (ii) for  $t < t'$  the closure of  $\Omega'_t$  is contained in  $\Omega'_{t'}$ , and
- (iii)  $\partial\Omega'_t$  tends to  $c = \partial\Omega'$  as  $t$  tends to 1.

Such a family may be obtained, for example, by uniformising the annulus obtained by removing a disc from  $\Omega'$ . Let  $\delta$  be a positive real number. Suppose that  $k < \delta$ . Let  $d$  be such that the Gaussian curvature of  $\Sigma'_d$  is greater than  $\delta$  in the weak sense. For  $t \in [0, 1]$  we define  $k_t \in [0, k]$  by:

$$k_t = t\delta.$$

For all  $t$ , let  $\Sigma''_t = (S, k_t)$  be the unique solution to the Plateau problem given by  $\hat{\mathbb{C}} \setminus \overline{\Omega}_t$  with constant Gaussian curvature equal to  $k_t$ .  $\Sigma''_0$  is a totally geodesic submanifold of  $\mathbb{H}^3$ . We may assume that  $\Omega_0$  is chosen sufficiently small so that  $\Sigma''_0$  is disjoint from  $\Sigma'_d$  in  $\mathbb{H}^3 \cup \partial_\infty \mathbb{H}^3$ . By completing the continuity argument as in the proof of Lemma 3.1, we see that  $\Sigma_k$  lies in the interior of  $\Sigma'_d$ . Since  $d$  may be made arbitrarily small, the result now follows.  $\square$

## 5 - Smoothness of the Foliation.

We have the following result:

### Lemma 5.1

*The foliation  $(\Sigma_k)_{k \in ]0, 1[}$  is smooth.*

**Proof:** We consider the quotient foliation in  $\mathbb{H}^3/\rho(\Gamma)$ . Thus every surface in the foliation is compact. Let  $k$  be a real number in  $]0, 1[$ . Let  $M$  be a field of symmetric positive definite matrices over  $S$ . For  $f$  a function over  $S$ , we define:

$$\Delta^M f = \text{Tr}(M^{-1} \text{Hess}(f)).$$

Consider now the following system of coupled partial differential equations:

$$\begin{aligned} \partial_t A &= f \text{Id} - \text{Hess}(f) - f A^2, \\ \Delta^A f + \text{Tr}(A - A^{-1})f &= -1, \end{aligned}$$

subject to the initial condition that  $A(0) = A_k$ . The second condition ensures that  $\text{Det}(A(t)) = k + t$  for all  $f$ . Let  $M$  be a  $2 \times 2$  symmetric matrix such that  $\text{Det}(M) = k$ . If  $\lambda \in \mathbb{R}$  be the lower of the two eigenvalues of  $M$ , then:

$$\text{Tr}(A - A^{-1}) = \frac{(k - 1)(\lambda^2 + k)}{\lambda k}.$$

Since  $\text{Det}(A_k) = k < 1$ , it follows that for small values of  $t$ :

$$\text{Tr}(A_t - A_t^{-1}) < 0.$$

Consequently, by the maximum principal, since  $S$  is compact, for small  $t$ , the operator  $\Delta^{A_t} + \text{Tr}(A_t - A_t^{-1})$  is injective. Any Laplacian acting on the space of functions over a compact manifold is of index zero, and thus, for small  $t$ , the operator  $\Delta^{A_t} + \text{Tr}(A_t - A_t^{-1})$  is surjective. Thus, for all  $t$ , there exists a unique solution  $f_t$  to the second equation. Using classical techniques of partial differential equations, we may thus smoothly solve this system for small values of  $t$ .

We now consider the mapping  $\Phi : S \times ]-\epsilon, \epsilon[ \rightarrow M$  and the vector field  $\mathbf{N} \in \Phi^*TM$  defined by the following system of coupled ordinary differential equations:

$$\begin{aligned}\partial\Phi &= f\mathbf{N}, \\ \nabla_{\partial_t}\mathbf{N} &= -\nabla^\Sigma f,\end{aligned}$$

subject to the condition that  $\Phi(\cdot, 0) = i_k$  and that  $\mathbf{N}(\cdot, 0)$  is the exterior unit normal vector field to  $\Sigma_k$ . This system of equations may be smoothly and uniquely solved for small values of  $t$ . Following the proof of Lemma 2.1, we see that, since they both satisfy the same differential equations with the same initial conditions, for all  $t$ , the field  $\mathbf{N}_t$  is equivalent to the exterior unit normal vector field to  $(S, \Phi_t)$ . Likewise, by Lemma 2.1, the Weingarten operator of  $(S, \Phi_t)$  coincides with  $A_t$ . Consequently, the surface  $(S, \Phi_t)$  is of constant Gaussian curvature equal to  $k + t$ , and thus, by uniqueness, it coincides with  $\Sigma_{k+t}$ .

It follows that the foliation is smooth near  $\Sigma_k$ , and since  $k \in ]0, 1[$  is arbitrary, the result now follows.  $\square$

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